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Time - dependent Ginzburg - Landau approach and application to superconductivity

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Abstract

A time dependent generalization of the Ginzburg -Landau Lagrangian is proposed. It contains two terms determining the time dependence and the four arbitrary scalar functions. Relevant equations, which coincide with equations following from the suitable Hamiltonian, are derived by a standard variational technique. These equations determine the energy conservation law and admit twofold time dependence which leads either to first or to second order time derivatives in Ginzburg - Landau equations. By introducing the gauge invariant potentials and choosing the gauge which differs slightly from the classical Lorentz one, the theory simplifies significantly. The results gained are discussed and compared to some earlier propositions. The presented approach, when reduced to a static one, is found to be in perfect agreement with that reported recently by the Koláček group. This indicates indirectly, that the equation with the first order time derivative seems to be more justified.

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1 Introduction

The Ginzburg - Landau (GL) free energy and the derived then consequently GL equations have proven to be very fruitful in the theory of superconductivity. Different situations, a lot of extensions, generalizations and applications to the different type of superconductors can be found in a huge and commonly known literature (c.f. [1] , [2]). But, perhaps as a natural consequence of the original article [3], the majority of papers and textbooks starts from a static version, when time is completely ignored.

The aim of the present paper is to correct this deficiency and to incorporate time in the frame of GL formalism, so that the obtained relations can be useful in the analysis of dynamical problems in the theory of superconductivity. The static reduction should then coincide with the known relations, issuing from the free energy concept.

There were different attempts to include time in the GL formalism. These attempts can be divided into two groups. In the papers of the first group, usually the time dependent formulation is postulated, starting a priori from the time-dependent Ginzburg - Landau (TDGL) equation $\gamma\hbar\psi_t = \left[(2m)^{-1}(\hbar\nabla)^2 + \alpha - \beta|\psi|^2\right]\psi$, as in [4] or similar. Its applicability to superconductive problems is later discussed and investigated. We are aware of a rich literature devoted to the mathematical aspects of this equation and with a proper choice of constants with its affinity to the nonlinear Schrödinger equation, see e.g. [5]. The second group, far less numerous, is formed by publications, in which the authors start by choosing some reasonable Lagrangian, as e.g.[6], [7]. There are also some papers, in which time is introduced from other premises e.g. [8].

We would like to point out that in the time dependence is involved also the problem of the scalar potential and charge density. Can the gauge be chosen such that scalar potential can vanish, and what is the role of a nonzero charge density then? In many textbooks considering static consequences of GL equations, the scalar potential is assumed to be zero without convincing arguments. Better insight into this problem can gained from references [9], [7], and references therein listing earlier contributions. In this paper we propose a modification of the GL formalism in order to include the time dependence starting from a suitable Lagrangian. Since the authors are convinced that there may be different approaches to the problem, in the first part there is proposed a general form of a GL-like Lagrangian, in which time is included in two different manners. Moreover, there are included four arbitrary scalar functions, which give additional freedom and assure some universality of considerations. In principle, the whole approach is gauge invariant. Next, the relevant modified GL equations are derived. These, however, are rather complicated and illegible, when as independent quantities are chosen the standard order parameter and potentials. Nevertheless, the formalism can be considerably simplified by the

introduction of so called gauge - invariant (g.i.) potentials. This concept is only partially new, since the first of these potentials is the commonly known gauge invariant phase.

The usefulness of this approach for superconductivity is discussed in the next sections. We show that our treatment in the static case is in agreement with the static approach reported in references[7], [9], [10],. More precisely, some particular static reduction of our general theory is in very good agreement with the references cited, with an accuracy to a single term, which exists in our theory, but probably was simplified in the cited papers. This result serves as an important hint concerning the form of TDGL. The discussed agreement indicates that the quantity, which has physical interpretation as charge density must be independent of scalar potential. A natural consequence is such reduction of TDGL that leads to the presence of the first order time derivative in GL equation [11], which however seems to be in contrast to the case considered in reference [6].

This last result automatically determines the final form of the TDGL-like Lagrangian and, as a consequence, the TDGL equations.

2 Time dependent Ginzburg - Landau approach

We propose the extended version of the Lagrangian density which includes a time dependence in the form

$$\mathcal{L} = \mathcal{L}_{GL} + \mathcal{L}_T, \tag{1}$$

$$\mathcal{L}_{GL} = \frac{1}{2}\left[(\mathbf{A}_t + \nabla\varphi)^2 - (\nabla\times\mathbf{A})^2\right] - W - V\left[(-i\nabla - \mathbf{A})\,\psi\right]\left[(i\nabla - \mathbf{A})\,\tilde{\psi}\right], \tag{2}$$

$$\mathcal{L}_T = M\left[i\left(\tilde{\psi}\partial_t\psi - \psi\partial_t\tilde{\psi}\right) - 2\psi\tilde{\psi}\varphi\right] + N\left[(i\partial_t - \varphi)\,\psi\right]\left[(-i\partial_t - \varphi)\,\tilde{\psi}\right]. \tag{3}$$

The canonical quantities here are: vector and scalar potentials $\mathbf{A}$ and φ, complex order parameters ψ and $\tilde{\psi}$, and the density of the charge carriers n. We assume that the functions W, V, M, N are algebraic real functions of two arguments: $|\psi|$ and n, i.e. $W = W(q,n)$ with.$q := |\psi|$, etc. and play a role of weight functions which can be fixed for concrete problems. Later we will impose the symmetry condition that $\tilde{\psi} = \psi^*$.

Choosing e.g. $W = |\psi|^2 + 2|\psi|^4$ and dropping the term $(\mathbf{A}_t + \nabla\varphi)^2$, it is seen that $\mathcal{L}_{GL}$ coincides with the standard, static version of GL free energy which can be found elsewhere, (e.g. [1], [2]). Some particular choices of W, V, M, N were reported in [12] and [6], announced also in[11] and in the static version discussed in numerous books and papers e.g. [1] [3], [10], . Thus the $\mathcal{L}_{GL}$ describes the static part of the Lagrangian density and $\mathcal{L}_T$, the supplement introducing a twofold time dependence in further field equations. We introduce canonical variable n describing the density of free charge carriers under influence of the lecture and paper [10]. Of course all these quantities depend on space coordinates x, y, z and time t.

The proposed Lagrangian density is gauge invariant with respect to the transformation

$$\{\mathbf{A} \Rightarrow \mathbf{A} - \nabla\chi,\ \varphi \Rightarrow \varphi + \chi_t,\ \psi \Rightarrow \psi\exp(i\chi)\,,\ \tilde{\psi} \Rightarrow \tilde{\psi}\exp(-i\chi)\,,\ \theta \Rightarrow \theta + \chi,\ n \Rightarrow n\}, \tag{4}$$

which will be evident shortly below (under assumption $\tilde{\psi} = \psi^*$).

The standard variational procedure leads to the system of rather complicated equations

$$\delta_{\mathbf{A}} : (\mathbf{A}_t + \nabla\varphi)_t + \nabla\times\nabla\times\mathbf{A} + V\left[i\left(\tilde{\psi}\nabla\psi - \psi\nabla\tilde{\psi}\right) + 2\mathbf{A}\psi\tilde{\psi}\right] = 0 \tag{5}$$

$$\delta_{\varphi} : -\Delta\varphi - \nabla(\mathbf{A}_t) - 2M\psi\tilde{\psi} + N\left[2\varphi\psi\tilde{\psi} + i\left(\psi\tilde{\psi}_t - \tilde{\psi}\psi_t\right)\right] = 0 \tag{6}$$

$$\delta_n : \left\{\begin{array}{c} W_n + V_n\left[(-i\nabla - \mathbf{A})\,\psi\right]\left[(i\nabla - \mathbf{A})\,\tilde{\psi}\right] - M_n\left[i\left(\tilde{\psi}\partial_t\psi - \psi\partial_t\tilde{\psi}\right) - 2\psi\tilde{\psi}\varphi\right] \\ -N_n\left[(i\partial_t - \varphi)\,\psi\right]\left[(-i\partial_t - \varphi)\,\tilde{\psi}\right] = 0 \end{array}\right. \tag{7}$$

$$\delta_{\tilde{\psi}} : \left\{\begin{array}{c} W_{\tilde{\psi}} - V_{\tilde{\psi}}\left[(i\nabla + \mathbf{A})\,\psi\right]\left[(i\nabla - \mathbf{A})\,\tilde{\psi}\right] - M_{\tilde{\psi}}\left[i\left(\tilde{\psi}\partial_t\psi - \psi\partial_t\tilde{\psi}\right) - 2\psi\tilde{\psi}\varphi\right] \\ +N_{\tilde{\psi}}\left[(i\partial_t - \varphi)\,\psi\right]\left[(i\partial_t + \varphi)\,\tilde{\psi}\right] + V\left[-\Delta\psi + i\left(\psi\nabla\cdot\mathbf{A} + 2\mathbf{A}\cdot\nabla\psi\right) + \mathbf{A}^2\psi\right] \\ +N\left[\partial_t^2\psi + i\left(\psi\partial_t\varphi + 2\varphi\partial_t\psi\right) - \varphi^2\psi\right] + 2M\left[\psi\varphi - i\partial_t\psi\right] \\ -\left(\nabla\psi - i\mathbf{A}\psi\right)\nabla V + \left(\partial_t\psi + i\varphi\psi\right)\partial_t N - iq\partial_t M = 0 \end{array}\right. \tag{8}$$

Equations (5) and (6) can be rewritten as nonhomogeneous wave equations for potentials

$$\Delta\mathbf{A}-\mathbf{A}_{tt}=V\left[i\left(\tilde{\psi}\nabla\psi-\psi\nabla\tilde{\psi}\right)+2\mathbf{A}\psi\tilde{\psi}\right]+\nabla\left(\nabla\cdot\mathbf{A}+\varphi_t\right) \tag{9}$$

$$\Delta\varphi-\varphi_{tt}=-2M\psi\tilde{\psi}+N\left[2\varphi\psi\tilde{\psi}+i\left(\psi\tilde{\psi}_t-\tilde{\psi}\psi_t\right)\right]-\left(\nabla\cdot\mathbf{A}+\varphi_t\right)_t \tag{10}$$

The last terms in both equations vanish, if the Lorentz gauge $\nabla\cdot\mathbf{A}+\varphi_t=0$ is assumed. Then the first terms on the right hand sides represent the current $\mathbf{j}$ and charge ρ densities, respectively. Thus

$$\mathbf{j}=V\left[i\left(\tilde{\psi}\nabla\psi-\psi\nabla\tilde{\psi}\right)+2\mathbf{A}\psi\tilde{\psi}\right] \tag{11}$$

$$\rho=2M\psi\tilde{\psi}-N\left[2\varphi\psi\tilde{\psi}+i\left(\psi\tilde{\psi}_t-\tilde{\psi}\psi_t\right)\right] \tag{12}$$

Assuming $\tilde{\psi}=\psi^*$ and calculating the imaginary part of $[(8)\exp(-i\theta)]$ one obtains immediately the continuity equation $\nabla\cdot\mathbf{j}+\rho_t=0$ with $\mathbf{j}$ and ρ given by (11) and (12), respectively.

Thus in the presented formalism we are able to include time dependence and four different types of weight functions. The physical interpretation will be evident later on. Here, it is worthwhile to remark that a uniform and time independent charge density of the background or lattice can be introduced by a suitable choice of M function.

As it was mentioned above, all derived relations are gauge invariant with respect to (4), although it is not seen directly. It can be done quite immediately defining new variables $\mathbf{F}$ and f which surely are gauge invariant by relations

$$\mathbf{F}:=\nabla\theta-\mathbf{A},\qquad f:=\varphi+\theta_t. \tag{13}$$

It is obvious that the first quantity $\mathbf{F}$ is the gauge invariant phase commonly introduced description different superconductive problems. The second one, f was to our knowledge never defined. Both quantities redefine the vector and scalar potentials and therefore we shall call them the gauge invariant potentials (g.i. potentials).

It appears that one can completely rewrite the Lagrangian density (1 - 3) using g.i. potentials according to (13). We have then

$$\mathcal{L}=\frac{1}{2}\left[\left(\mathbf{F}_t-\nabla f\right)^2-\left(\nabla\times\mathbf{F}\right)^2\right]-W-\left[q^2\mathbf{F}^2+\left(\nabla q\right)^2\right]V+N\left(q^2f^2+{q_t}^2\right)-2Mq^2f, \tag{14}$$

where $q^2=\psi\tilde{\psi}$ and the remaining symbols have the meaning as before.

Lagrangian density (1) - (3) is now automatically gauge invariant since it is defined by the gauge invariant quantities. Considering now $\mathbf{F}$,f,q,n as new variables we obtain the system of equations

$$\delta_{\mathbf{F}}:\left(\mathbf{F}_t-\nabla f\right)_t+\nabla\times\nabla\times\mathbf{F}+2q^2V\mathbf{F}=0, \tag{15}$$

$$\delta_f:\Delta f-\nabla\cdot\mathbf{F}_t+2q^2\left(M-Nf\right)=0, \tag{16}$$

$$\delta_n:W_n+V_n\left[q^2\mathbf{F}^2+\left(\nabla q\right)^2\right]-N_n\left(q^2f^2+{q_t}^2\right)+2q^2M_nf=0, \tag{17}$$

$$\delta_q:\left\{\begin{array}{c} V\Delta q-Nq_{tt}-\frac{1}{2}W_q-q\left(\mathbf{F}^2V-f^2N\right)-\frac{1}{2}\left[\mathbf{F}^2q^2-\left(\nabla q\right)^2\right]V_q-N_nn_tq_t \\ +\frac{1}{2}\left[q^2f^2-\left(q_t\right)^2\right]N_q+V_n\left[\nabla n\cdot\nabla q\right]-fq\left(2M+qM_q\right)=0 \end{array}\right. \tag{18}$$

Rewriting the first two equations as before, we have the nonhomogeneous wave equations

$$\Delta\mathbf{F}-\mathbf{F}_{tt}=2q^2V\mathbf{F}+\nabla\left(\nabla\cdot\mathbf{F}-f_t\right), \tag{19}$$

$$\Delta f-f_{tt}=-2q^2\left(M-Nf\right)+\left(\nabla\cdot\mathbf{F}-f_t\right)_t, \tag{20}$$

with the current and charge density defined as

$$\mathbf{j}=2qV\mathbf{F}, \tag{21}$$

$$\rho=2q^2\left(M-Nf\right). \tag{22}$$

Now it is easy to check the continuity equations $\nabla\cdot\mathbf{j}+\rho_t=0$. Indeed, by (15) and (16)

$$\nabla\cdot\mathbf{j}=\nabla\cdot(2qV\mathbf{F})=-\nabla\mathbf{F}_{tt}+\Delta f_t=-2\left[q^2\left(M-Nf\right)\right]_t=-\rho_t.$$

Two possible charge density components are given by (22) and it is only a question of a concrete physical problem which of them can be neglected, putting the relevant constant equal to zero. (The are no inevitable arguments to assume $N=0$; e.g. in paper [13] there is considered the configuration when $\rho\sim\varphi$.)

As in the case where on the vector and scalar potentials the Lorentz gauge could be imposed, now it can be also done formulating the Lorentz-like gauge imposed on g.i. potentials in form of condition

$$\nabla\cdot\mathbf{F}-f_t=0. \tag{23}$$

This requirement itself is also gauge invariant since it contains gauge invariant quantities, in contrast to the Lorentz gauge condition for $\mathbf{A}$ and f , which is not invariant with respect gauge (4), unless $\Delta\theta-\theta_{tt}=0$. It is obvious, since in the language of standard potentials equation (23) reduces to $\nabla\cdot A+\varphi_t=\Delta\theta-\theta_{tt}$.

When the static solutions are considered $f\equiv\varphi$ and the system of equations (15) - (23) reduces to

$$\Delta\mathbf{F}-2q^2V\mathbf{F}=0, \tag{24}$$

$$\Delta f+2q^2\left(M-Nf\right)=0, \tag{25}$$

$$W_n+V_n\left[q^2\mathbf{F}^2+(\nabla q)^2\right]-N_nq^2f^2+2q^2M_nf=0, \tag{26}$$

$$\begin{aligned} V\Delta q-\tfrac{1}{2}\left[W_q+\left(\mathbf{F}^2q^2-(\nabla q)^2\right)V_q-q^2f^2N_q\right]-q\left(\mathbf{F}^2V-f^2N\right)+V_n\left[\nabla n\cdot\nabla q\right]\\ =fq\left(2M+qM_q\right), \end{aligned} \tag{27}$$

where we assumed the static reduction of the Lorentz gauge for q.i. potentials $\nabla\cdot\mathbf{F}=\mathbf{0}$. The current and charge density definitions (15) and (16), respectively, remain unchanged.

Defining the canonical momenta $\mathbf{p}_F,p_f,p_q,p_n$ associated with variables $\mathbf{F},f,q,n$ in a standard form, we have

$$\mathbf{p}_F=\mathbf{F}_t-\nabla f,\ \ p_f=0,\ \ p_q=2Nq_t,\ \ p_n=0. \tag{28}$$

Therefore the Hamiltonian density $\mathcal{H}=\mathbf{F}_t\cdot\mathbf{p}_F+q_tp_q-\mathcal{L}$ is

$$\mathcal{H}=\frac{1}{2}\left[\mathbf{p}_F^2+(\nabla\times\mathbf{F})^2\right]+\nabla f\cdot\mathbf{p}_F+\frac{(p_q)^2}{4N}+W+\left[q^2\mathbf{F}^2+(\nabla q)^2\right]V+2Mq^2f-Nq^2f^2 \tag{29}$$

where the whole term $(p_q)^2/4N$ drops out if $N=0$ and then also $p_q=0$. The standard Hamilton equations reproduce (28) and equations (15 - 18), as should be. Substituting (28) into (14), the energy density is

$$\begin{aligned}\mathcal{E}=\frac{1}{2}\left[\mathbf{F}_t^2-(\nabla f)^2+(\nabla\times\mathbf{F})^2\right]+W+\left[q^2\mathbf{F}^2+(\nabla q)^2\right]V+\\ +2Mq^2f+N\left[(q_t)^2-f^2\right].\end{aligned} \tag{30}$$

It is natural to ask about the conservation relations. Tedious, though simple calculations lead to the equation

$$\begin{aligned}\partial_t\left[\frac{1}{2}\left[(\mathbf{F}_t)^2+(rot\mathbf{F})^2-(\nabla f)^2\right]+W+\left[q^2F^2+(\nabla q)^2\right]V+2Mq^2f+N\left[(q_t)^2-f^2\right]\right]\\ =\nabla\cdot\left(\mathbf{F}_t\times rot\mathbf{F}+\left(F_t-\nabla f\right)f_t+2Vq_t\nabla q\right).\end{aligned} \tag{31}$$

Till now, still the real functions W,V,M,N are completely arbitrary and some of them can vanish. Observe however, that the right hand side of (31) i.e. flux depends only on V. Moreover, meanwhile we did not impose the Lorentz-like gauge (23) and also in particular cases the variable n can vanish simplifying all previous relations.

Admitting however the Lorentz-like gauge (23) system of equations (15) - (20) becomes

$$\Delta\mathbf{F} - \mathbf{F}_{tt} = j = 2q^2 V\mathbf{F}, \tag{32}$$
$$\Delta f - f_{tt} = -\rho = -2q^2 (M - Nf), \tag{33}$$
$$W_n + V_n \left[q^2\mathbf{F}^2 + (\nabla q)^2\right] - N_n \left(q^2 f^2 + {q_t}^2\right) + 2q^2 M_n f = 0, \tag{34}$$
$$\begin{aligned} V\Delta q - Nq_{tt} &= \frac{1}{2}W_q + q\left(\mathbf{F}^2 V - f^2 N\right) + \frac{1}{2}\left[\mathbf{F}^2 q^2 - (\nabla q)^2\right] V_q \\ &- \frac{1}{2}\left[q^2 f^2 - (q_t)^2\right] N_q - V_n \left[\nabla n \nabla q\right] + N_n n_t q_t + fq\left(2M + qM_q\right), \end{aligned} \tag{35}$$

and in the next paragraph we shall discuss some particular situations due to the specific choice of W, V, M, and N functions.

3 Elementary reductions

As a first example we consider the system of equations which follows from Lagrangian (1) - (3) and from the relevant equations when the variable n does not appear at all, $V = N = 1$ and M vanishes. Moreover, assuming $W(q) = q^2 + \beta q^4/2$, from (32) - (35) and (31) we obtain the system of equations:

$$\mathcal{L} = \frac{1}{2}\left[(\mathbf{F}_t - \nabla f)^2 - (\nabla \times \mathbf{F})^2\right] - W - \left[q^2\mathbf{F}^2 + (\nabla q)^2\right] + \left(q^2 f^2 + {q_t}^2\right) \tag{36}$$
$$\Delta\mathbf{F} - \mathbf{F}_{tt} = j = 2q^2\mathbf{F} \tag{37}$$
$$\Delta f - f_{tt} = -\rho = 2q^2 f, \tag{38}$$
$$\Delta q - q_{tt} = \frac{1}{2}W_q + q\left(F^2 - f^2\right) \tag{39}$$
$$\begin{aligned} &\partial_t\left[\frac{1}{2}\left[(\mathbf{F}_t)^2 + (rot\mathbf{F})^2 - (\nabla f)^2\right] + W + \left[q^2\mathbf{F}^2 + (\nabla q)^2\right] + \left[(q_t)^2 - f^2\right]\right] \\ &= \nabla \cdot \left(\mathbf{F}_t \times rot\mathbf{F} + (F_t - \nabla f) f_t + 2q_t \nabla q\right) \end{aligned} \tag{40}$$

One can find elsewhere Its static version, but when the time dependence in included, it was discussed only in a few publications e.g. [6] [12],. To our knowledge it has no broader application in superconductivity, probably because of two rather serious insufficiencies: the charge is proportional to the scalar potentials (see r.h.s. of (38) and the second order derivatives with respect to time q_{tt} in equation (39) appears. Rewriting this system in a standard language of A, φ, ψ, we will have the second derivative ψ_{tt}, which seems to be at least strange when the time dependent Ginzburg - Landau equation (or Gross - Pitaevskiii eqn.) are compared with the classical Schrödinger equation.

In order to derive the system of equations with the first order derivative ψ_t in relevant GL equation, as e.g. in paper [4] one can consider a dual situation when $V = M = 1$ and N vanishes (also $n \equiv 0$). The system (32) - (35) and (31) reduces then to

$$\mathcal{L} = \frac{1}{2}\left[(\mathbf{F}_t - \nabla f)^2 - (\nabla \times \mathbf{F})^2\right] - W - \left[q^2\mathbf{F}^2 + (\nabla q)^2\right] - 2q^2 f, \tag{41}$$
$$\Delta\mathbf{F} - \mathbf{F}_{tt} = j = 2q^2\mathbf{F}, \tag{42}$$
$$\Delta f - f_{tt} = -\rho = -2q^2, \tag{43}$$
$$\Delta q = \frac{1}{2}W_q + q\mathbf{F}^2 + 2fq, \tag{44}$$
$$\begin{aligned} &\partial_t\left[\frac{1}{2}\left[(F_t)^2 + (rot\mathbf{F})^2 - (\nabla f)^2\right] + W + \left[q^2\mathbf{F}^2 + (\nabla q)^2\right] + 2q^2 f\right] \\ &= \nabla \cdot \left(\mathbf{F}_t \times rot\mathbf{F} + (\mathbf{F}_t - \nabla f) f_t + 2q_t \nabla q\right). \end{aligned} \tag{45}$$

In contrast to the previous choice, now in language of g.i. potentials, equation (44) does not contain the time derivative, but it is present when the system is written by means of $\mathbf{A}, \varphi, \psi$ variables. In fact, instead of (44), equation (8) reduces to form

$$i\partial_t\psi = \frac{1}{2}\left(i\nabla + \mathbf{A}\right)^2\psi + \frac{1}{2}W_{\tilde{\psi}} + \psi\varphi. \tag{46}$$

A comparison of both above discussed examples speaks in favor of the choice $N = 0$, since it leads to the first order in time GL equation.

In the next paragraph we consider a further argument in favor of that choice.

4 Application to the superconductivity

In a series of papers [7] [10], there was considered the Bardeen phenomenological extension of the static GL theory, mainly in order to include the energy of electric field and to demonstrate an important role of the scalar potential which is very often neglected. Unfortunately this theory is completely static, starting from the time independent free energy. Neglecting the unimportant details and in symbols (signature) adopted in present paper the free energy discussed there has the form

$$\mathcal{L} = \frac{1}{2}\left[-\left(\nabla\varphi\right)^2 + \left(\nabla\times\mathbf{A}\right)^2\right] + U + w + \frac{n}{4m}\left|\left(i\nabla + \mathbf{A}\right)\psi\right|^2 + \varphi\rho \tag{47}$$

There is introduced a new variable n − the density of the total charge, while ψ is reserved for the order parameter. The function w includes the Gorter-Casimir corrections and $w = -\left[\frac{1}{4}T_c^2\left|\psi\right|^2 - \frac{1}{2}T^2\sqrt{1 - \left|\psi\right|^2}\right]\gamma$. Essential for further consideration is that $\gamma = \gamma\left(n\right)$ and thus $w = w\left(\left|\psi\right|, n\right)$. Moreover, the effective mass of the carriers also depends on density of charge i.e. $m = m\left(n\right)$. The function U introduced in (47) described the effect of the screening on the Thomas - Fermi length and what is important $U = U\left(n\right)$ and thus $\partial U/\partial n = C\rho$, where C is constant. From the context of the paper it follows also that authors assume $\rho = en$.

According to the authors, variations with respect A, φ, ψ and n variables lead to the system of equations

$$\nabla\times\nabla\times\mathbf{A} = \nabla\times H = -\frac{n}{2m}\operatorname{Re}\left(\tilde{\psi}\left(i\nabla + \mathbf{A}\right)\psi\right) \tag{48}$$

$$-\Delta\varphi = \rho \tag{49}$$

$$e\varphi = -C\rho - w_n - \frac{1}{4}\frac{1}{m}\left(1 - \frac{d\left(\ln m\right)}{d\left(\ln n\right)}\right)\left|\left(i\nabla + \mathbf{A}\right)\psi\right|^2 \tag{50}$$

$$\frac{n}{4m}\left(i\nabla + \mathbf{A}\right)^2\psi = -\psi w_{\left|\psi\right|^2} \tag{51}$$

Accepting without discussion the physical aspect and phenomenological justification of the free energy according to expression (47) we shall show that one can propose an expression for the Lagrangian which includes time dependence and next to derive the self-consistent time dependent field equations which quite completely reduce to the system (48) - (51), when only static effects are considered. "Quite completely" means here that - up to the terms $0\left(\nabla\left(n/m\right)\right)$. The reasons that the terms proportional to $\nabla\left(n/m\right)$ can be neglected in equation (51) was not specified. A proposition of a proper Lagrangian reduces in practice to a suitable choice of four functions W, V, M and N .in our considerations.

Let assume that

$$V = n/\left(4m\right), \;\; W = \alpha n^2/2 + w, \;\; M = \alpha n/\left(2\left|\psi\right|^2\right), \; N = 0, \tag{52}$$

where still $m = m\left(n\right)$, $w = w\left(\left|\psi\right|^2, n\right)$ and α is constant. Lagrangian (1) takes the form

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\left[\left(\mathbf{A}_t + \nabla\varphi\right)^2 - \left(\nabla\times\mathbf{A}\right)^2\right] - \left(\alpha n^2/2 + w\right) - \frac{n}{4m}\left|\left(i\nabla + \mathbf{A}\right)\psi\right|^2 \\ + \frac{\alpha n}{2\left|\psi\right|^2}\left[i\left(\tilde{\psi}\partial_t\psi - \psi\partial_t\tilde{\psi}\right) - 2\psi\tilde{\psi}\varphi\right].\end{aligned} \tag{53}$$

It differs from the free energy (47) practically only by time dependent terms. Consequently, the proper field equations (5) - (8) are

$$\delta_{\mathbf{A}}:\Delta\mathbf{A}-\mathbf{A}_{tt}-\frac{n}{4m}\left[i\left(\tilde{\psi}\nabla\psi-\psi\nabla\tilde{\psi}\right)+2\mathbf{A}\psi\tilde{\psi}\right]=\nabla\left(\nabla\cdot\mathbf{A}+\varphi_t\right), \tag{54}$$

$$\delta_{\varphi}:\Delta\varphi-\varphi_{tt}=-\alpha n-\left(\nabla\mathbf{A}+\varphi_t\right)_t\ , \tag{55}$$

$$\delta_n: a\left(n+\varphi\right)+w_n+\frac{1}{4}\left(\frac{n}{m}\right)_n\left|\left(i\nabla+\mathbf{A}\right)\psi\right|^2-\frac{\alpha}{2\left|\psi\right|^2}i\left(\tilde{\psi}\partial_t\psi-\psi\partial_t\tilde{\psi}\right)=0, \tag{56}$$

$$\delta_{\tilde{\psi}}:\left|\psi\right|^2 w_{|\psi|^2}+\frac{n}{4m}\tilde{\psi}\left(i\nabla+\mathbf{A}\right)^2\psi-\frac{1}{4}\left(\tilde{\psi}\nabla\psi-i\mathbf{A}\left|\psi\right|^2\right)\nabla\left(\frac{n}{m}\right)-i\frac{\alpha}{2}n_t=0. \tag{57}$$

Assuming a Lorentz gauge $\nabla\mathbf{A}+\varphi_t=0$ and observing that $\left(n/m\right)_n=\left[1-d\left(\ln m\right)/d\left(\ln n\right)\right]/m$, it is seen that static version of the system (54) - (57) is quite equivalent to system (48) - (51). The difference is only because of the term $-\frac{1}{4}\left(\tilde{\psi}\nabla\psi-i\mathbf{A}\left|\psi\right|^2\right)\nabla\left(n/m\right)$ in the equation (57), which in (51) was probably omitted or simplified.

We would underline that the time dependent system (54) - (57) is self-consistent with all consequences which were mentioned in the previous paragraphs. The "prototype" of the GL equation (56) contains only the first order time derivative.

Rewriting all equations in the language of the gauge invariant potentials $\mathbf{F}$ and f, the whole theory becomes much simpler, particularly when the Lorentz-like gauge is adopted. Substituting (52) to (14) - (18) we obtain Lagrangian

$$\mathcal{L}=\frac{1}{2}\left[\left(\mathbf{F}_t-\nabla f\right)^2-\left(\nabla\times\mathbf{F}\right)^2\right]-\frac{\alpha n^2}{2}-w-\frac{n}{4m}\left[q^2\mathbf{F}^2+\left(\nabla q\right)^2\right]-\alpha nf, \tag{58}$$

and relevant field equations

$$\delta_{\mathbf{F}}:\Delta\mathbf{F}-\mathbf{F}_{tt}-\frac{n}{2m}q^2\mathbf{F}=-\nabla\left(\nabla\cdot\mathbf{F}-f_t\right)_t, \tag{59}$$

$$\delta_f:\Delta f-f_{tt}+\alpha n=\left(\nabla\cdot\mathbf{F}-f_t\right)_t, \tag{60}$$

$$\delta_n:\alpha n+w_n+\left(\frac{n}{4m}\right)_n\left[q^2\mathbf{F}^2+\left(\nabla q\right)^2\right]+af=0, \tag{61}$$

$$\delta_q:\frac{n}{m}\Delta q-2w_q-\frac{n}{m}qF^2+\left(\frac{n}{m}\right)_n\nabla n\cdot\nabla q=0, \tag{62}$$

together with a conservation law (Poyinting like theorem)

$$\begin{aligned}\partial_t&\left[\frac{1}{2}\left[\left(F_t\right)^2+\left(rot\mathbf{F}\right)^2-\left(\nabla f\right)^2\right]+w+\frac{n}{4m}\left[q^2\mathbf{F}^2+\left(\nabla q\right)^2\right]+\frac{n\alpha}{2}\left(n+2f\right)\right]\\&=\nabla\cdot\left(\mathbf{F}_t\times rot\mathbf{F}+\left(\mathbf{F}_t-\nabla f\right)f_t+\frac{n}{2m}q_t\nabla q\right).\end{aligned} \tag{63}$$

It is seen that the system (58) - (63) is much simpler than those expressed by the standard potentials, especially when the condition $\nabla\cdot\mathbf{F}-f_t=0$ is imposed. Moreover, the time derivatives do not appear at all in equations (61) and (62), suggesting to use the static solutions as a first step when the perturbation technique will be applied for the solution of a dynamic problem.

Repeating the calculations of this paragraph in case when $N\neq0$ and it is e.g.$N=\beta n/2\left|\psi\right|^2$, where β is constant, the static versions of equations (54) and (57) do not change but the static version of equations (55) and (56) take a form

$$\Delta\varphi=-n\left(\alpha+\beta\varphi\right);\quad a\left(n+\varphi\right)+w_n+\frac{1}{4}\left(\frac{n}{m}\right)_n\left|\left(i\nabla+\mathbf{A}\right)\psi\right|^2-\frac{\beta}{2}\varphi^2=0. \tag{64}$$

It is seen that the agreement of this system with the system (48) - (51), which was proposed on a base of physical premises can be reached only when $\beta=0$, i.e. when in the starting Lagrangian (1) -(3) N function vanishes. The proposed term, proportional to M, will be the sole one which includes time dependence.

5 Conclusion

We proposed a general and extended formula for the Lagrangian in spirit of the Ginzburg - Landau formalism, augmented by the time-dependent terms of two types. Moreover, a few arbitrary weight scalar functions introduced in this Lagrangian allow us to derive a family of the self-consistent time - dependent extended Ginzburg - Landau equations which coincide with the relevant Hamilton equations. Among them there are families with first and second time derivatives of the order parameter functions. The whole theory and equations can be simplified when it is rewritten by means of the gauge - invariant potentials. As an example of the application, the static version of GL equations reported in [7] [10], are derived and verified as the reduction of a full dynamical theory.

This reproducibility seems to indicate that for the correct dynamic theory, the scalar function N discussed in the text should vanish. This means that then only the first order time derivative will be present in the time dependent version of GL formalism.

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